

Irreducible Characters for Algebraic Groups in Characteristic Three

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Abstract

In this note, we determine the irreducible characters for the simple algebraic groups of type A_5 over an algebraically closed field K of characteristic 3, by using a theorem of Xi Nanhua [7] and the Matlab software. In order to obtain higher speed than in [9, 10] we modify the algorithm to compute the irreducible characters.

Keywords: Irreducible character, Semisimple algebraic group, Composition factor

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The determination of all irreducible characters is a big theme in the modular representations of algebraic groups and related finite groups of Lie type. But so far only a little is known concerning it in the case when the characteristic of the base field is less than the Coxeter number.

Gilkey-Seitz gave an algorithm to compute part of characters of $L(\lambda)$'s with $\lambda \in X_1(T)$ for G being of type G_2 , F_4 , E_6 , E_7 and E_8 in characteristic 2 and even in larger primes in [3]. Dowd and Sin gave all characters of $L(\lambda)$'s with $\lambda \in X_1(T)$ for all groups of rank less than or equal to 4 in characteristic 2 in [2]. They got their results by using the “standard” Gilkey-Seitz algorithm and computer. L. Scott et al. computes the characters for A_4 when $p = 5, p = 7$ by computing the maximal submodule in a baby Verma module [11]. Anders Buch and Niels Lauritzen also obtain this result for A_4 when $p = 5$ with Jantzen’s sum formula [12].

An element $\mathfrak{r}_{(p^n-1)\rho-\lambda} \in \mathfrak{u}_n^-$ for each irreducible module $L(\lambda)$ with $\lambda \in X_n(T)$ was defined in [5, §39.1, p. 304] and [7, p. 239]. This element could be used in constructing a certain basis for $L(\lambda)$, computing $\dim L(\lambda)$, and determining $\text{ch}(L(\lambda))$. In this way, Xu and Ye, Ye and Zhou determined all irreducible characters for the special linear groups $SL(5, K)$, $SL(6, K)$ and $SL(7, K)$, the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field K of characteristic 2 in [6, 8] and for the special orthogonal group $SO(7, K)$ and the symplectic group $Sp(6, K)$ over an algebraically closed field K of characteristic 3 in [9, 10]. However, it need so much time to compute the irreducible characters when the characteristic is bigger than 3 that this become to be impossible mission. So we must find out faster algorithm to finish this work. In the present note, we shall work out all irreducible characters for the simple algebraic groups of type A_5 over an algebraically closed field K of characteristic 3 and introduce how to obtain faster speed with modified algorithm. Of course we will explain why our results should be right. We shall freely use the notations in [9] without further comments.

1 PRELIMINARIES

Let G be the simple algebraic group of type A_5 over an algebraically closed field K of characteristic 3. Take a Borel subgroup B and a maximal torus T of G with $T \subset B$. Let $X(T)$ be the character group of T , which is also called the weight lattice of G with respect to T . Let $R \subset X(T)$ be the root system associated to (G, T) , and choose a positive root system R_+ in such a way that $-R_+$ corresponds to B . Let

$$S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$

be the set of simple roots of G such that

$$R_+ = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_{ij} = \alpha_i + \cdots + \alpha_j, 1 \leq i < j \leq 5\}.$$

Let $\omega_i (1 \leq i \leq 5)$ be the fundamental weights of G such that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$, the Kronecker delta, and denote by $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ the weight $\lambda = \lambda_1\omega_1 + \lambda_2\omega_2 + \lambda_3\omega_3 + \lambda_4\omega_4 + \lambda_5\omega_5$ with $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{Z}$, the integer ring. Then the dominant weight set is as follows:

$$X(T)_+ = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in X(T) \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0\}.$$

Let $W = N_G(T)/T$ be the Weyl group and let W_3 be the affine Weyl group of G . It is well-known that for $\lambda \in X(T)_+$, $H^0(\lambda)$ is the induced G -module from the 1-dimensional B -module K_λ which contains a unique irreducible G -submodule $L(\lambda)$ of the highest weight λ . In this way, $X(T)_+$ parameterizes the finite-dimensional irreducible G -modules. We set $\text{ch}(\lambda) = \text{ch}(H^0(\lambda))$ and $\text{ch}_3(\lambda) = \text{ch}(L(\lambda))$ for all $\lambda \in X(T)_+$. Moreover, $\text{ch}(\lambda)$ is given by the Weyl character formula, and for $\lambda \in X(T)_+$, we have

$$\text{ch}(\lambda) = \frac{\sum_{w \in W} \det(w) e(w(\lambda + \rho))}{\sum_{w \in W} \det(w) e(w\rho)}.$$

For $\lambda = (a, b, c, d, e) \in X_1(T)$, we have

$$\begin{aligned} \dim H^0(a, b, c, d, e) = & \frac{1}{2^8 3^3 5} (a+1)(b+1)(c+1)(d+1)(e+1)(a+b+2) \\ & (b+c+2)(c+d+2)(d+e+2)(a+b+c+3) \\ & (b+c+d+3)(c+d+e+3)(a+b+c+d+4) \\ & (b+c+d+e+4)(a+b+c+d+e+5), \end{aligned}$$

Let F^n be the n -th Frobenius morphism of G with $G_n \subset G$ the scheme-theoretic kernel of F^n . Let $V^{[n]}$ be the Frobenius twist for any G -module V . It is well-known that $V^{[n]}$ is trivial regarding as a G_n -module. Moreover, any G -module M has such a form if the action of G_n on M is trivial. Let

$$X_n(T) = \{(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in X(T)_+ \mid \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 < 3^n\}.$$

Then the irreducible G -modules $L(\lambda)$'s with $\lambda \in X_n(T)$ remain irreducible regarded as the G_n -modules. On the other hand, any irreducible G_n -module is isomorphic to exactly one of them.

For $\lambda \in X(T)_+$, we have the unique decomposition

$$\lambda = \lambda^0 + 3^n \lambda^1 \quad \text{with} \quad \lambda^0 \in X_n(T), \lambda^1 \in X(T)_+.$$

Then the Steinberg tensor product theorem tells us that

$$L(\lambda) \cong L(\lambda^0) \otimes L(\lambda^1)^{[n]}.$$

Therefore we can determine all the characters $\text{ch}_3(\lambda)$ with $\lambda \in X(T)_+$ by using the Steinberg tensor product theorem, provided that all the characters $\text{ch}_3(\lambda)$ with $\lambda \in X_1(T)$ are known.

Recall the strong linkage principle in [1]. We define a strong linkage relation $\mu \uparrow \lambda$ in $X_+(T)$ if $L(\mu)$ occurs as a composition factor in $H^0(\lambda)$.

Then $H^0(\lambda)$ is irreducible when λ is a minimal weight in $X(T)_+$ with respect to the partial ordering determined by the strong linkage relations.

Let \mathfrak{g} be the simple Lie algebra over \mathbb{C} which has the same type as G , and \mathfrak{U} the universal enveloping algebra of \mathfrak{g} . Let $e_\alpha, f_\alpha, h_i (\alpha \in R_+, i = 1, 2, 3, 4, 5)$ be a Chevalley basis of \mathfrak{g} . We also denote $e_{\alpha_I}, f_{\alpha_I}$ by e_I, f_I , respectively, where $I \in \mathcal{A} = \{1, 2, 3, 4, 5, 12, 23, 34, 45, 13, 24, 35, 14, 25, 15\}$. The Kostant \mathbb{Z} -form $\mathfrak{U}_{\mathbb{Z}}$ of \mathfrak{U} is the \mathbb{Z} -subalgebra of \mathfrak{U} generated by the elements $e_\alpha^{(k)} := e_\alpha^k/k!, f_\alpha^{(k)} := f_\alpha^k/k!$ for $\alpha \in R_+$ and $k \in \mathbb{Z}_+$. Set

$$\binom{h_i + c}{k} := \frac{(h_i + c)(h_i + c - 1) \cdots (h_i + c - k + 1)}{k!}.$$

Then $\binom{h_i + c}{k} \in \mathfrak{U}_{\mathbb{Z}}$, for $i = 1, 2, 3, 4, 5, c \in \mathbb{Z}, k \in \mathbb{Z}_+$. Define $\mathfrak{U}_k := \mathfrak{U}_{\mathbb{Z}} \otimes K$ and call \mathfrak{U}_k the hyperalgebra over K associated to \mathfrak{g} . Let $\mathfrak{U}_k^+, \mathfrak{U}_k^-, \mathfrak{U}_k^0$ be the positive part, negative part, zero part of \mathfrak{U}_k , respectively. They are generated by $e_\alpha^{(k)}, f_\alpha^{(k)}$ and $\binom{h_i}{k}$, respectively. By abuse of notations, the images in \mathfrak{U}_k of $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i + c}{k}$, etc. will be denoted by the same notations, respectively. The algebra \mathfrak{U}_k is a Hopf algebra, and \mathfrak{U}_k has a triangular decomposition $\mathfrak{U}_k = \mathfrak{U}_k^- \mathfrak{U}_k^0 \mathfrak{U}_k^+$. Given a positive integer n , let \mathfrak{U}_n be the subalgebra of \mathfrak{U}_k generated by the elements $e_\alpha^{(k)}, f_\alpha^{(k)}, \binom{h_i}{k}$ for $\alpha \in R_+, i = 1, 2, 3, 4, 5$ and $0 \leq k < 3^n$. In particular, $\mathfrak{U} = \mathfrak{U}_1$ is precisely the restricted enveloping algebra of \mathfrak{g} . Denote by $\mathfrak{U}_n^+, \mathfrak{U}_n^-, \mathfrak{U}_n^0$ the positive part, negative part, zero part of \mathfrak{U}_n , respectively. Then we have also a triangular decomposition $\mathfrak{U}_n = \mathfrak{U}_n^- \mathfrak{U}_n^0 \mathfrak{U}_n^+$. Given an ordering in R_+ , it is known that the PBW-type bases for \mathfrak{U}_k resp. for \mathfrak{U}_n have the form of

$$\prod_{\alpha \in R_+} f_\alpha^{(a_\alpha)} \prod_{i=1}^5 \binom{h_i}{b_i} \prod_{\alpha \in R_+} e_\alpha^{(c_\alpha)}$$

with $a_\alpha, b_i, c_\alpha \in \mathbb{Z}_+$ resp. with $0 \leq a_\alpha, b_i, c_\alpha < 3^n$.

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in X_n(T)$. We set $\lambda_I = \sum_{i \in I} \lambda_i$ for $I \in \mathcal{A}$, here each element I is also viewed as a certain set of simple roots. Following [5] and [7], we define an elements \mathfrak{x}_λ in \mathfrak{U}_n^- by

$$\mathfrak{x}_\lambda = \frac{f_1^{(\lambda_1)} f_2^{(\lambda_{12})} f_3^{(\lambda_{13})} f_4^{(\lambda_{14})} f_5^{(\lambda_{15})} f_1^{(\lambda_2)} f_2^{(\lambda_{23})} f_3^{(\lambda_{24})} f_4^{(\lambda_{25})}}{f_1^{(\lambda_3)} f_2^{(\lambda_{34})} f_3^{(\lambda_{35})} f_1^{(\lambda_4)} f_2^{(\lambda_{45})} f_1^{(\lambda_5)}}.$$

As a special case of [7, Theorems 6.5 and 6.7], we have Theorem 1 Assume that \mathfrak{g} is a simple Lie algebra of the simple algebraic group of type A_5 over an

algebraically closed field K of characteristic 3. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \in X_n(T)$.

- (i) The element \mathfrak{r}_λ lies in \mathfrak{U}_n^- .
- (ii) Let \mathfrak{J}_λ be the left ideal of \mathfrak{U}_k generated by the elements $e_i^{(k)}, \binom{h_i}{k} - \binom{\langle \lambda, \alpha_i^\vee \rangle}{k}, f_i^{(k_i)}$ ($i = 1, 2, 3, 4, 5, k \geq 1, k_i \geq 3^n$) and the elements $f \in \mathfrak{U}_n^-$ with $f\mathfrak{r}_{(3^n-1)\rho-\lambda} = 0$. Then $\mathfrak{U}_k/\mathfrak{J}_\lambda \cong L(\lambda)$ (Note that $L(\lambda)$ has a \mathfrak{U}_k -module structure, which is irreducible).
- (iii) As a \mathfrak{U}_n^- -module, $L(\lambda)$ is isomorphic to $\mathfrak{U}_n^-\mathfrak{r}_{(3^n-1)\rho-\lambda}$.

By abuse of notations, the images in $\mathfrak{U}_k/\mathfrak{J}_\lambda \cong L(\lambda)$ of $f_i^{(k_i)}$ and $f_I^{(k_I)}$ will be denoted by the same notations. We shall use this theorem to computer the multiplicities of the weight spaces for all the dominant weight of $L(\lambda)$, to compute $\dim L(\lambda)$, and to determine $\text{ch}(L(\lambda)) = \text{ch}_3(\lambda)$ ($\lambda \in X_1(T)$) in this note, when G is the simple algebraic group of type A_5 .

2 CHARACTERS OF THE IRREDUCIBLE MODULES OF G

From now on we shall assume that $n = 1$. Denote by V^* the dual module of V , then we have by the duality that $\text{ch}H^0(\lambda)^* = \text{ch}(-w_0\lambda)$, and $\text{ch}L(\lambda)^* = \text{ch}_3(-w_0\lambda)$. Furthermore, the elements f_I ($I \in \mathcal{A}$) satisfy the following commutator relations:

$$\begin{aligned} f_1f_2 &= f_2f_1 + f_{12}, & f_2f_3 &= f_3f_2 + f_{23}, \\ f_3f_4 &= f_4f_3 + f_{34}, & f_{12}f_3 &= f_3f_{12} + f_{123}, \\ f_{23}f_4 &= f_4f_{23} + f_{234}, & f_1f_{23} &= f_{23}f_1 + f_{123}, \\ f_2f_{34} &= f_{34}f_2 + f_{234}, & f_1f_{234} &= f_{234}f_1 + f_{1234}, \\ f_{12}f_{34} &= f_{34}f_{12} + f_{1234}, & f_{123}f_4 &= f_4f_{123} + f_{1234}, \\ f_I f_{I'} &= f_{I'} f_I & \text{for all the other } I, I' \in \mathcal{A}. \end{aligned}$$

Now we can obtain our main theorems. Let $e(\nu) = \sum_{w \in W_\nu} w(\nu)$ be the sum of weights of the W -orbit of ν for all $\nu \in X(T)_+$. It is well-known that $\{\text{ch}(\nu) | \nu \in X(T)_+\}$, $\{\text{ch}_3(\nu) | \nu \in X(T)_+\}$ and $\{e(\nu) | \nu \in X(T)_+\}$ form bases of $\mathbb{Z}[X(T)]^W$, the W -invariant subring of $\mathbb{Z}[X(T)]$, respectively. According to the Weyl character formula and the Freudenthal multiplicity formula, we get a change of basis matrix $A = (a_{\lambda\nu})_{\lambda, \nu \in X(T)_+}$ from $\{e(\nu) | \nu \in X(T)_+\}$ to

$\{\text{ch}(\nu)|\nu \in X(T)_+\}$, which is a triangular matrix with 1 on its diagonal, i.e.

$$\text{ch}(\lambda) = \sum_{\nu \prec \lambda, \nu \in X(T)_+} a_{\lambda\nu} e(\nu)$$

with $a_{\lambda\lambda} = 1$ (cf. [12]). Based on our computation, we get another change of basis matrix $B = (b_{\lambda\nu})_{\lambda, \nu \in X(T)_+}$ from $\{e(\nu)|\nu \in X(T)_+\}$ to $\{\text{ch}_3(\nu)|\nu \in X(T)_+\}$, which is also a triangular matrix with 1 on its diagonal.

Let us mention our computation of B more detailed. First of all, we compute $\mathfrak{x}_{2\rho-\lambda}$ for any $\lambda \in X_1(T)$. It is well known that for each dominant weight ν of $H^0(\lambda)$, $\beta = \lambda - \nu$ can be expressed in terms of sum of positive roots, and there exist many ways to do so. Each way corresponds to an element $f_\beta \mathfrak{x}_{2\rho-\lambda}$ in \mathfrak{U}_n . Then we compute various $f_\beta \mathfrak{x}_{2\rho-\lambda}$. Note that each $f_\beta \mathfrak{x}_{2\rho-\lambda}$ can be written as a linear combination of the basis elements of \mathfrak{U}_n with non-negative integer coefficients, and the typical images of all non-zero $f_\beta \mathfrak{x}_{2\rho-\lambda}$'s generate the weight space $L(\lambda)_\nu$ of the irreducible submodule $L(\lambda)$ of $H^0(\lambda)$. Therefore, we can easily determine the dimension of $L(\lambda)_\nu$, provided that we compute the rank of the set of all these non-zero $f_\beta \mathfrak{x}_{2\rho-\lambda}$'s. It can be reduced to compute the rank of a corresponding matrix. Finally, we obtain the formal character of $L(\lambda)$, which can be written as a linear combination of $e(\nu)$'s with non-negative integer coefficients. That is

$$\text{ch}_3(\lambda) = \sum_{\nu \prec \lambda, \nu \in X(T)_+} b_{\lambda\nu} e(\nu)$$

with $b_{\lambda\lambda} = 1$. In this way, we get the second matrix B .

For example, we assume that G is the simple algebraic group of type A_5 and $\lambda = (2, 1, 2, 1, 2)$.

It is easy to see that

$$\mathfrak{x} = \mathfrak{x}_{2\rho-\lambda} = \mathfrak{x}_{(01010)} = f_2 f_1 f_3 f_2 f_4^{(2)} f_3^{(2)} f_2 f_1 f_5^{(2)} f_4^{(2)} f_3 f_2.$$

For $\nu = (3, 0, 1, 2, 2)$, we have $\lambda - \nu = (-1, 1, 1, -1, 0) = \alpha_2 + \alpha_3$. First we compute each of the set $SS_\nu = \{f_2 f_3 \mathfrak{x}, f_{23} \mathfrak{x}\}$. Then we compute the rank of the set SS_ν , which is equal to 2. So we have $\dim L(2, 1, 2, 1, 2)_{(3,0,1,2,2)} = 2$. For $\mu = (2, 0, 1, 1, 3)$, we have $\lambda - \mu = (0, 1, 1, 0, -1) = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4$. We compute each of the set $SS_\mu = \{f_1 f_2 f_2 f_3 f_3 f_4 \mathfrak{x}, f_1 f_2 f_2 f_3 f_{34} \mathfrak{x}, f_1 f_2 f_3 f_{234} \mathfrak{x}, f_1 f_2 f_{23} f_{34} \mathfrak{x}, f_1 f_{23} f_{234} \mathfrak{x}, f_1 f_{23} f_{23} f_4 \mathfrak{x}, f_{12} f_2 f_3 f_3 f_4 \mathfrak{x}, f_{12} f_2 f_3 f_{34} \mathfrak{x}, f_{12} f_{23} f_{34} \mathfrak{x}, f_{12} f_{234} f_3 \mathfrak{x}, f_{12} f_{23} f_3 f_4 \mathfrak{x}, f_{123} f_{23} f_4 \mathfrak{x}, f_{123} f_2 f_3 f_4 \mathfrak{x}, f_{123} f_2 f_{34} \mathfrak{x}, f_{123} f_{234} \mathfrak{x},$

$f_{1234}f_{23}\mathfrak{x}, f_{1234}f_2f_3\mathfrak{x}\}$, and then we compute the rank of the set SS_μ , which is equal to 13. So we have $\dim L(2, 1, 2, 1, 2)_{(2,0,1,1,3)} = 13$. By this methods, we can calculate all multiplicity $b_{\lambda\nu}$. Finally, we obtain the formal character of irreducible module $\text{ch}_3(2, 1, 2, 1, 2)$.

When λ lies in $X(T)_+$ but not in $X_1(T)$, we can also compute the formal character $\text{ch}_3(\lambda)$ by using the Steinberg tensor product theorem. For $\lambda \in X(T)_+$, we have the unique decomposition

$$\lambda = \lambda^0 + 3\lambda^1 \quad \text{with} \quad \lambda^0 \in X_1(T), \lambda^1 \in X(T)_+.$$

Then the Steinberg tensor product theorem tells us that

$$\text{ch}_3(\lambda) = \text{ch}_3(\lambda^0) \cdot \text{ch}_3(3\lambda^1).$$

Therefore, we can determine all characters $\text{ch}_3(\lambda)$ with $\lambda \in X(T)_+$, provided that all characters $\text{ch}_3(\lambda)$ with $\lambda \in X_1(T)$ are known. For example, when $\lambda = (0, 2, 0, 0, 3)$, we have

$$\begin{aligned} \text{ch}_3(0, 2, 0, 0, 3) &= \text{ch}_3(0, 2, 0, 0, 0) \cdot \text{ch}_3(0, 0, 0, 0, 3) \\ &= (e(0, 2, 0, 0, 0) + e(1, 0, 1, 0, 0) + e(0, 0, 0, 1, 0)) \cdot e(0, 0, 0, 0, 3) \\ &= e(0, 2, 0, 0, 3) + e(2, 0, 0, 0, 1) + e(1, 0, 1, 0, 3) + e(1, 1, 0, 0, 2) \\ &\quad + e(0, 1, 0, 0, 1) + e(0, 0, 0, 1, 3) + e(0, 0, 1, 0, 2). \end{aligned}$$

Therefore, from the two matrices A, B , we can easily get the third change of basis matrix $D = AB^{-1}$ from $\{\text{ch}_3(\nu) | \nu \in X(T)_+\}$ to $\{\text{ch}(\nu) | \nu \in X(T)_+\}$, which is still a triangular matrix with 1 on its diagonal. The matrix D gives the decomposition patterns of various $H^0(\lambda)$ with $\lambda \in X(T)_+$.

We list the matrix D in the attached tables. In all these tables, the left column indicates λ 's. For two weight $\nu \prec \lambda \in X(T)_+$, the number $d_{\lambda\nu}$ in tables is just the multiplicity of composition factors $[H^0(\lambda) : L(\nu)]$.

3 FASTER ALGORITHM

In our earlier paper [9, 10], we compute the multiplicity $b_{\lambda\nu}$ one by one for a fixed weight λ . However, noticing that some information computing $b_{\lambda\nu}$ may be useful to compute $b_{\lambda\mu}$ for $\nu \preccurlyeq \mu$. So we compute all possible f_β , such that $SS_\lambda = \{f_\beta\mathfrak{x}\}$ spanning to the whole $L(\lambda)$, firstly. Then we compute $SS_\lambda = \{f_\beta\mathfrak{x}\}$ in some ordering: if $f_\beta = f_{\beta_1}f_{\beta_2}$, then we first obtain

$y_1 = f_{\beta_2}\mathfrak{x}$, save this result and compute $y_2 = f_{\beta}\mathfrak{x} = f_{\beta_1}y_1$ instead of computing $f_{\beta}\mathfrak{x} = f_{\beta_1}f_{\beta_2}\mathfrak{x}$, directly. In fact we only need compute $f_{\beta}y$, for some positive root β and $y \in SS_{\lambda}$ in one step.

For example, suppose to compute $\{f_3f_4\mathfrak{x}, f_{23}f_4\mathfrak{x}\}$, we can compute $y_1 = f_4\mathfrak{x}$ at the first step, and then compute $y_2 = f_3y_1, y_3 = f_{23}y_1$. In this way, we can avoid much repeated work.

4 WHY THESE RESULTS ARE TRUE

In order to obtain the results the computer must work several days. So we must be careful to avoid error. There are facts to verify the results.

At firstly, we compute the dimension of weight space, then by Sternberg tensor formula and Weyl formula we obtain the decomposition pattern of $H^0(\lambda)$. At last checking all the data we find that

1). Symmetry of dimension of weight space. Checking the results the two equations are satisfied:

$$\begin{aligned} \dim L(\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1)_{(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)} &= \dim L(\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1)_{(\mu_5, \mu_4, \mu_3, \mu_2, \mu_1)}, \\ \dim L(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)_{(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)} &= \dim L(\lambda_5, \lambda_4, \lambda_3, \lambda_2, \lambda_1)_{(\mu_5, \mu_4, \mu_3, \mu_2, \mu_1)}. \end{aligned}$$

2). Symmetry of composition factors. From the $H^0(\lambda)$'s decomposition patterns, the following equations are hold:

$$\begin{aligned} &[H^0(\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1) : L(\mu_1, \mu_2, \mu_3, \mu_4, \mu_5)] \\ &= [H^0(\lambda_1, \lambda_2, \lambda_3, \lambda_2, \lambda_1) : L(\mu_5, \mu_4, \mu_3, \mu_2, \mu_1)]. \end{aligned}$$

3). Positivity of multiplicity of composition factors. All the multiplicity of composition factors we obtained are nonnegative.

4). Linkage principle is hold. If the multiplicity of composition factors $[H^0(\lambda) : L(\nu)] \neq 0$, then we have $\mu \uparrow \lambda$.

From the representation theory of algebraic groups, all the above results should be hold, so the computational data is compatible with the theory. Hence we may accept these results.

5 MAIN RESULTS

Theorem When $G = SL(6, K)$, let $\Lambda = \{(2, 2, 2, 2, 2), (1, 2, 2, 2, 2), (1, 2, 2, 2, 2), (2, 1, 0, 2, 2), (2, 2, 0, 1, 2), (2, 2, 2, 0, 1), (1, 0, 2, 2, 2), (2, 0, 1, 2, 2), (2, 2, 1, 0, 2),$

$(0, 2, 2, 2, 2), (2, 2, 2, 2, 0), (0, 1, 2, 2, 2), (2, 2, 2, 1, 0) \in X_1(T)$. Then $H^0(\lambda)$ is an irreducible G -module for all $\lambda \in \Lambda$ and the decomposition patterns of $H^0(\lambda)$ for all $\lambda \in X_1(T) \setminus \Lambda$ are listed in Table 1-9.

Remark: The table should be read as following. We list the weights in the first collum and write the multiplicity of composition factors as the others elements of tables. For example, from the third row in table 1, we obtain 00200 0 1 1, this mean

$$\text{ch}(0, 0, 2, 0, 0) = 0 \cdot \text{ch}_3(0, 0, 0, 0, 0) + 1 \cdot \text{ch}_3(1, 0, 0, 0, 1) + 1 \cdot \text{ch}_3(0, 0, 2, 0, 0).$$

According to the symmetry of A_5 , we need not list all results. For example, we can obtain the decomposition pattern of $H^0(0, 0, 2, 0, 1)$ from table 2:

$$\text{ch}(0, 0, 2, 0, 1) = \text{ch}_3(0, 0, 2, 0, 1) + \text{ch}_3(1, 0, 0, 0, 2).$$

So we also have

$$\text{ch}(1, 0, 2, 0, 0) = \text{ch}_3(1, 0, 2, 0, 0) + \text{ch}_3(2, 0, 0, 0, 1).$$

Table 1

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00000 1
10001 1 1
00200 0 1 1
01002 0 1 0 1
20010 0 1 0 0 1
00111 1 1 1 1 0 1
11100 1 1 1 0 1 0 1
00030 1 0 0 0 0 1 0 1
00103 0 0 0 1 0 1 0 0 1
03000 1 0 0 0 0 0 1 0 0 1
30100 0 0 0 0 1 0 1 0 0 0 1
11011 2 2 1 1 1 1 1 0 0 0 0 1
11003 2 1 0 1 0 1 0 0 1 0 0 1 1
30011 2 1 0 0 1 0 1 0 0 0 1 1 0 1
00014 0 0 1 0 0 1 0 1 1 0 0 0 0 0 1
41000 0 0 1 0 0 0 1 0 0 1 1 0 0 0 0 1
30003 3 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1
02020 2 0 1 0 0 1 1 1 0 1 0 1 0 0 0 0 1
10112 3 1 1 1 0 2 1 0 1 0 0 1 1 1 0 0 0 0 1
21101 3 1 1 0 1 1 2 0 0 0 1 1 0 1 0 0 0 0 1
10031 1 0 0 0 0 1 0 1 0 0 0 0 0 0 0 0 1 0 1
13001 1 0 0 0 0 0 1 0 0 1 0 0 0 0 0 0 0 0 1
00400 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 0 1
02004 2 1 1 0 0 1 0 1 1 0 0 1 1 0 1 0 0 1 0 0 0 1
40020 2 0 1 0 0 0 1 0 0 1 1 1 0 1 0 1 0 1 0 0 0 1
10023 2 1 0 0 0 1 1 1 1 0 0 0 1 0 1 0 0 0 1 0 1
32001 2 0 1 0 0 1 1 1 0 0 1 1 0 0 1 1 0 1 0 0 0 1
01121 2 0 1 0 0 1 1 1 1 0 0 0 1 1 0 0 0 0 1 0 1
12110 2 0 1 0 0 1 1 1 0 0 1 0 1 0 1 0 0 0 0 0 1
20202 6 1 1 0 0 1 1 1 0 0 0 0 1 1 1 0 0 1 0 1 0 0 1
00311 0 0 1 0 0 0 1 0 0 1 0 0 0 0 0 0 0 1 0 0 0 1 0 0 1
11300 0 0 1 0 0 1 0 1 0 0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 0 1
01113 3 1 1 0 1 1 2 1 1 1 1 0 2 0 1 0 0 1 1 0 1 0 1 0 1 0 1
31110 3 1 1 0 0 2 1 1 0 1 1 0 2 0 1 0 1 0 1 0 1 0 1 0 0 0 1
40004 3 2 0 0 0 0 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0 0 0 0 1
00303 0 0 0 0 1 0 1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1
30300 0 0 0 1 0 1 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1
01032 1 0 0 0 0 0 1 0 0 0 0 0 0 1 0 0 0 0 0 2 0 0 0 0 1 0 1 0 0 0 1
23010 1 0 0 0 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 1 0 2 0 0 0 1 0 1 0 0 0 1
11211 6 0 2 0 0 2 2 1 0 1 0 1 1 1 1 0 0 2 1 1 1 1 1 0 0 1 1 1 1 0 0 1
11203 8 1 2 0 1 1 3 0 0 2 1 1 2 1 0 0 1 1 1 1 1 0 1 0 1 0 1 0 1 0 0 1
30211 8 1 2 1 0 3 1 2 1 0 0 1 1 2 0 0 1 1 1 1 1 0 0 1 0 1 0 1 0 1 0 1
30203 10 2 4 0 0 2 2 1 1 1 1 2 2 1 1 2 1 1 1 1 0 1 1 1 1 0 0 0 0 1 0 0 1
21212 10 3 7 1 1 3 3 1 3 1 2 2 1 1 1 2 1 1 2 1 1 1 1 1 1 1 1 1 0 1 1 1 1

```

Table 2

00001	1	10002	1
12000	1 1	20100	0 1
31000	0 1 1	00201	1 0 1
00120	1 0 0 1	20011	1 1 0 1
11020	2 1 0 1 1	20003	1 0 0 1 1
00104	0 0 0 1 0 1	11101	1 1 1 1 0 1
30020	2 1 1 0 1 0 1	03001	0 0 0 0 0 1 1
22001	1 1 1 0 0 0 0 1	30101	0 1 0 1 0 1 0 1
11004	2 0 0 1 1 1 0 0 1	02110	0 0 1 1 0 1 1 0 1
10121	1 0 0 1 1 0 0 0 0 1	10202	1 0 1 1 1 1 0 0 0 1
21110	2 1 1 1 1 0 1 1 0 0 1	41001	0 0 1 0 0 1 1 1 0 0 1
30004	3 0 0 0 1 0 1 0 1 0 0 1	01300	0 0 1 0 0 0 0 0 1 0 0 1
20300	0 0 0 1 0 0 0 0 0 0 1 0 1	10040	0 0 0 0 0 0 0 0 0 1 0 0 1
10113	2 1 0 1 1 1 0 0 0 1 1 0 0 0 1	40110	1 0 1 1 0 1 1 1 1 0 1 0 0 1
13010	0 0 0 0 0 0 0 1 0 0 1 0 0 0 1	01211	0 0 1 1 0 1 1 0 1 1 0 1 0 0 1
10032	0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 1	01130	0 0 0 1 1 0 0 0 0 1 0 0 1 0 1 1
20211	2 0 0 1 1 0 1 1 0 1 1 0 1 0 0 0 1	01203	0 1 0 1 1 1 1 0 0 1 0 0 0 0 1 0 1
01122	1 1 0 0 1 0 0 0 1 1 0 0 0 1 0 1 0 1	00320	0 0 0 0 0 0 0 1 0 0 0 0 1 0 0 1 1 0 1
20203	4 1 1 0 1 0 1 1 1 1 0 1 0 1 0 0 1 0 1	01041	0 0 0 0 1 0 0 0 0 1 0 1 1 0 1 1 1 0 1
00312	0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 1	11220	0 0 1 1 1 1 1 0 1 1 0 1 1 0 1 1 0 1 0 1
11212	4 2 1 1 1 0 1 1 1 1 1 0 1 1 1 1 1 1 1 1	30220	1 0 2 1 1 1 0 0 1 1 0 0 1 1 0 0 0 0 1 1
		21221	2 1 3 1 2 1 1 0 1 1 0 1 1 1 1 1 1 1 1 1

Table 3

10010	1
01100	1 1
01011	1 1 1
01003	0 0 1 1
50000	0 1 0 0 1
00112	0 1 1 1 0 1
00031	0 0 0 0 0 1 1
11012	1 1 1 1 0 1 0 1
00023	0 1 0 1 0 1 1 0 1
30012	1 0 0 0 0 0 0 1 0 1
02021	0 1 0 0 0 1 1 1 0 0 1
21102	1 1 0 0 0 1 0 1 0 1 0 1
02013	1 1 0 1 0 1 1 1 1 0 1 0 1
12200	0 1 0 0 0 0 0 0 0 0 0 0 1
13002	0 0 0 0 0 0 0 0 0 0 1 0 0 1
40021	1 1 0 0 1 0 0 1 0 1 1 0 0 0 0 1
31200	0 1 1 0 1 0 0 0 0 0 0 0 1 0 0 1
32002	0 1 0 0 1 1 0 0 0 1 0 1 0 0 1 0 0 1
12111	0 2 0 0 0 1 0 1 0 1 1 1 0 1 1 0 0 0 1
40013	2 0 0 0 0 0 0 1 0 1 1 0 1 0 0 1 0 0 0 1
23100	0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0 0 0 1
12103	1 2 0 0 0 1 0 1 0 1 1 1 1 0 1 0 0 0 1 0 0 1
31111	1 3 1 1 1 2 1 1 0 2 1 1 0 1 1 1 1 0 0 0 1
31103	2 4 0 1 1 2 1 1 1 2 1 1 1 0 1 1 1 0 1 1 1
23011	0 2 0 0 1 1 0 0 0 1 0 1 0 1 2 0 1 1 1 0 1 0 1
23003	0 3 0 0 1 1 0 0 0 1 0 1 0 0 2 0 0 1 1 0 0 1 1 1 1
22112	3 8 1 1 3 2 1 1 1 2 2 1 1 1 2 1 1 1 2 1 1 1 1

Table 4

10210	1	21021	1	02102	1	22010	1
02221	1 1	21013	1 1	40102	1 1	10122	0 1
		12022	1 1 1	22120	1 1 1	20212	1 1 1

Table 5

	10100	1
	10011	1 1
	10003	0 1 1
	01101	1 1 0 1
	00202	0 1 1 1 1
	20012	1 1 1 0 0 1
	50001	0 0 0 1 0 0 1
	00040	0 0 0 0 1 0 0 1
10012	1	
01102	1 1	
50002	0 1 1	
02201	0 1 0 1	
40201	1 1 1 1 1	
12210	0 1 0 1 0 1	
24001	0 0 0 1 1 0 1	
31210	1 2 0 1 1 1 0 1	
23110	0 2 1 1 1 1 1 1	
22300	0 1 0 0 0 1 0 1 1 1	
22211	1 2 1 1 1 1 1 1 1 1 1	
	10100	1
	10011	1 1
	10003	0 1 1
	01101	1 1 0 1
	00202	0 1 1 1 1
	20012	1 1 1 0 0 1
	50001	0 0 0 1 0 0 1
	00040	0 0 0 0 1 0 0 1
	11102	1 1 1 1 1 1 0 0 1
	02200	0 0 0 1 0 0 0 0 0 1
	03002	0 0 0 0 0 0 0 0 0 1 0 1
	30102	1 0 0 0 0 1 0 0 1 0 0 1
	02111	0 0 0 1 1 1 0 0 1 1 1 0 1
	40200	0 1 0 1 0 0 1 0 0 1 0 0 0 1
	41002	0 0 0 1 1 0 1 0 1 0 1 1 0 0 1
	02030	0 0 1 0 1 1 0 1 0 0 0 0 1 0 0 1
	02103	1 0 1 0 1 1 0 0 1 0 1 0 1 0 0 0 1
	40111	1 1 1 1 1 1 1 0 1 1 1 1 1 1 0 0 1
	24000	0 0 0 0 0 0 0 0 0 1 0 0 0 1 0 0 0 0 1
	40030	1 0 1 0 0 1 0 0 0 0 0 0 1 0 0 1 0 1 0 1
	40103	2 0 1 0 1 1 0 0 1 0 1 1 1 0 1 0 1 1 0 0 1
	12120	0 0 1 1 1 1 0 0 1 1 1 1 1 0 0 1 0 0 0 0 1
	31120	1 1 2 2 2 1 0 1 1 1 1 1 1 1 1 0 1 0 1 0 1 1
	23020	0 0 0 2 1 0 1 0 1 1 2 1 0 1 1 0 0 0 1 0 0 1 1 1
	22121	3 2 3 3 2 1 1 1 1 1 2 1 2 1 1 1 1 1 0 1 1 1 1 1

Table 6

12010	1	02101	1	01012	1	20101	1	20002	1
10212	1 1	01202	1 1	12201	0 1	01220	0 1	10201	1 1
10131	0 1 1	01040	0 1 1	31201	1 1 1	01204	1 1 1	01210	0 1 1
20221	1 1 1 1	21220	1 1 1 1	23101	0 1 1 1	01042	0 1 1 1	02220	0 1 1 1
				22202	1 1 1 1 1	21222	1 1 1 1 1	12221	1 1 0 1 1

Table 7

	00002	1
	21000	0 1
	00210	1 0 1
	20020	1 1 0 1
	12001	1 1 0 0 1
	31001	0 1 0 0 1 1
	11110	1 1 1 1 1 0 1
	20004	1 0 0 1 0 0 0 1
	10300	0 0 1 0 0 0 1 0 1
	03010	0 0 0 0 1 0 1 0 0 1
	30110	1 1 0 1 1 1 1 0 0 0 1
	10211	1 0 1 1 1 0 1 0 1 0 0 1
	10130	0 0 0 1 0 0 0 0 0 0 0 1 1
	10203	1 1 0 1 1 0 0 1 0 0 0 1 0 1
	10041	0 0 0 0 0 0 0 1 0 0 1 1 1 1
	01212	0 1 0 1 1 0 1 0 1 1 0 1 0 1 0 1
	20220	1 0 0 1 1 0 1 0 1 0 1 1 1 0 0 0 1
	01131	0 0 0 1 0 0 0 1 1 0 0 1 1 1 1 0 1
	00321	0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 0 1 1
	11221	2 1 1 1 1 0 1 1 2 1 1 1 1 1 1 1 1 1 1
	00010	1
	02000	1 1
	40000	0 1 1
	01020	1 1 0 1
	01004	0 0 0 1 1
	00121	0 0 0 1 0 1
	00113	0 1 0 1 1 1 1
	11021	1 1 0 1 0 1 0 1
	00032	0 0 0 0 0 1 1 0 1
	11013	2 1 0 1 1 1 1 1 0 1
	30021	2 1 1 0 0 0 0 1 0 0 1
	21200	0 1 1 1 0 0 0 0 0 0 0 1
	22002	1 1 1 0 0 0 0 0 0 0 0 0 1
	30013	3 0 0 0 0 0 0 1 0 1 1 0 0 1
	13100	0 0 0 0 0 0 0 0 0 0 0 1 0 0 1
	21111	2 1 1 1 0 1 0 1 0 0 1 1 1 0 0 1
	02022	1 1 0 0 0 1 1 1 1 1 0 0 0 0 0 1
	21103	4 1 1 0 0 1 1 1 0 1 1 0 1 1 0 1 0 1
	13011	0 0 1 0 0 0 0 0 0 0 0 0 1 1 0 1 1 0 0 1
	13003	1 0 1 0 0 0 0 0 0 0 0 0 1 0 0 1 0 1 1 1
	12112	4 3 3 1 0 1 1 1 0 1 1 1 1 1 1 2 1 1 1 1

Table 8

00100	1
00011	1 1
11000	1 0 1
00003	0 1 0 1
30000	0 0 1 0 1
10020	1 1 1 0 0 1
02001	1 1 1 0 0 0 1
01110	1 1 1 0 0 1 1 1
10004	0 1 0 1 0 1 0 0 1
40001	0 0 1 0 1 0 1 0 0 1
00300	0 0 0 0 0 0 0 1 0 0 1
00211	0 1 0 1 0 1 1 1 0 0 1 1
11200	0 0 1 0 1 1 1 1 0 0 1 0 1
20021	1 1 1 1 1 1 1 0 0 0 0 0 0 0 1
12002	1 1 1 1 1 1 0 1 0 0 0 0 0 0 0 1
00130	0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 1
00203	0 1 1 1 0 1 1 0 1 0 0 1 0 0 0 0 0 1
03100	0 0 0 0 0 0 1 0 0 0 0 0 1 0 0 0 0 0 1
30200	0 1 1 0 1 1 1 0 0 1 0 0 1 0 0 0 0 0 1
20013	1 1 0 1 0 1 0 0 1 0 0 0 0 1 0 0 0 0 0 1
31002	1 0 1 0 1 0 1 0 1 0 0 1 0 0 0 0 1 0 0 0 0 0 1
11111	1 1 1 1 1 1 2 2 1 0 0 1 1 1 1 1 0 0 0 0 0 0 1
11030	0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 1 0 0 0 0 0 1 1
03011	0 0 0 0 1 0 1 0 0 0 0 0 1 0 1 0 0 1 0 0 0 1 0 1
11103	2 1 1 2 1 1 1 0 1 0 0 1 0 1 1 0 1 0 0 1 0 1 0 0 1
30111	2 1 1 1 2 1 1 0 0 1 0 0 1 1 1 0 0 0 1 0 1 1 0 0 0 1
00041	0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 1 1 0 0 0 0 0 0 0 1
14000	0 0 0 0 0 0 0 0 0 0 0 1 0 1 0 0 0 0 0 1 1 0 0 0 0 0 0 0 1
30030	1 0 0 1 1 1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 0 0 1 0 0 1
03003	1 0 0 1 1 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 1 0 1 1 0 0 0 0 1
30103	3 0 0 1 1 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 0 0 1 0 0 0 0 1
02112	1 0 1 1 3 1 1 0 0 0 1 1 1 1 1 0 1 1 0 1 0 2 0 1 1 0 0 0 0 1 0 1
21120	1 1 0 3 1 1 1 0 0 0 1 1 1 1 1 1 1 0 0 1 0 1 2 1 0 0 1 0 0 1 0 0 0 1
02031	0 0 0 1 1 1 0 0 1 0 1 1 0 1 0 1 1 0 0 1 0 1 1 0 0 0 1 0 0 0 0 1 0 1
13020	0 0 0 1 1 0 1 0 0 1 1 0 1 0 1 0 0 1 1 0 1 1 0 1 0 0 0 1 0 0 0 0 1 0 1
12121	3 2 2 4 4 2 2 1 1 1 2 1 1 1 1 1 1 1 1 1 1 3 1 1 1 1 0 0 1 1 1 1 1 1 1

Table 9

00122	1	01010	1	10101	1	00022	1
22100	0 1	21012	1 1	20102	1 1	02012	1 1
11022	1 0 1	12021	0 1 1	02120	0 1 1	40012	0 1 1
22011	0 1 0 1	12013	1 1 1 1	02104	1 1 1 1	12102	0 1 0 1
30022	0 0 1 0 1	31021	1 1 1 0 1	40120	1 1 1 0 1	31102	1 1 1 1 1
22003	0 0 0 1 0 1	31013	2 1 1 1 1 1	40104	2 1 1 1 1 1	22200	0 0 0 0 0 1
21112	1 1 1 1 1 1 1	22022	3 1 2 1 1 1 1	22122	3 1 2 1 1 1 1	23002	0 0 0 1 1 0 1
						22111	1 1 1 1 1 1 1 1

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